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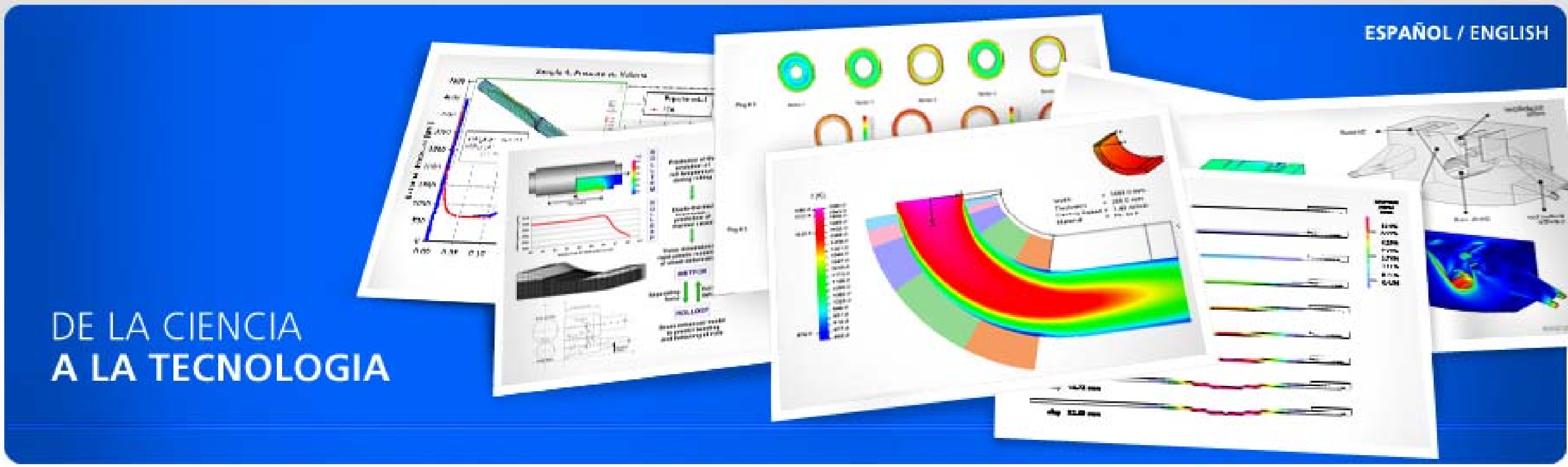

FEM in Heat Transfer
Part 2

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CONTENTS

Part 1

- Introduction to heat transfer
- Heat transfer equations
- Non-dimensional numbers
- The finite element method in heat transfer
- Boundary conditions: natural convection, forced convection
- Boundary conditions: radiation, boiling
- Examples on conduction-convection heat transfer problems

Part 2

- Boundary conditions: review
- Boundary conditions: condensation
- Time integration
- Non-linear equations: Picard method, Newton-Raphson method, BFGS
- Examples on transitory thermal problems

CONTENTS

Boundary conditions: review

- Dirichlet boundary conditions $T = T^{\mathrm{sup}}$ $\quad \forall(\underline{x}, t) \in \partial\Gamma_T \times \partial t$
- Newmann boundary conditions $q_n^* = -k \frac{\partial T}{\partial n} = Q_s(T) \quad \forall(\underline{x}, t)$ $\partial\Gamma_q \times \partial t$
  - where $\underline{\mathbf{n}}$ is the normal vector output to the domain surfac Q_s is positive when the heat output Ω.
- Mixt boundary conditions. Newton cooling law

$$q_n^* = -k \frac{\partial T}{\partial n} = h(T)\,(T - T_{amb}) \qquad \forall(\underline{x}, t) \in \partial\Gamma_c \times \partial t$$

Boundary conditions: review

- Radiation boundary conditions

$$q_n^* = -k \frac{\partial T}{\partial n} = \sigma F \epsilon \left(T^4 - T^4_{medio} \right) \quad \forall(\underline{x}, t) \in \partial\Gamma_r \times \partial t$$

where σ is the Stefan-Boltzmann constant $= 5.6697 \times 10^{-8} \frac{\mathrm{W}}{\mathrm{m^2 \cdot K^4}}$; ϵ is the emisitivy (nondimensional), F is the shape factor or vision factor (nondimensional)

- Robin boundary conditions

$$q_n^* = -k \frac{\partial T}{\partial n} = \hat{h}(T) \left(T - T_{amb} \right) \quad \forall(\underline{x}, t) \in \partial\Gamma_\sigma \times \partial t$$

$$\hat{h} = \hat{h}_{conv} + \hat{h}_{rad}$$

$$\hat{h}_{conv} = \textit{experimental, literature, etc.}$$

$$\hat{h}_{rad} = \sigma F \epsilon \frac{T^4 - T^4_{medio}}{T - T_{amb}}$$

Boundary conditions: review

Natural and Forced Convection

$$Nu = \left\{ A_1 + \frac{A_2\, Ra^{n_1}}{\left[1 + \left(A_3 / \mathrm{Pr}\right)^{n_2}\right]^{n_3}} \right\}^{n_4}$$

$$Nu = \frac{h\,L}{k_f}$$

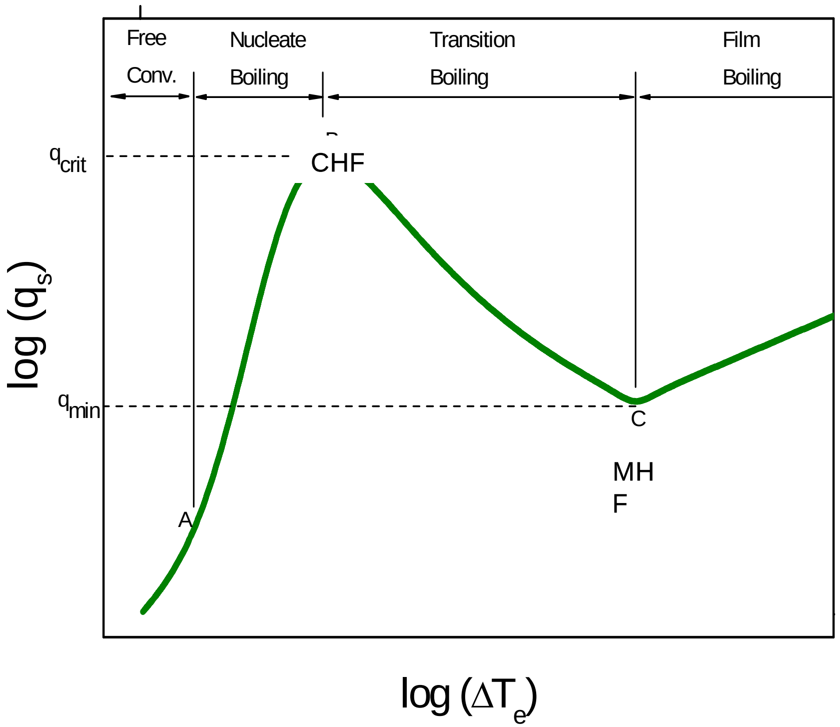
$$\mathrm{Pr} = \frac{\mu / \rho}{k / \rho C p} = \frac{\mu C p}{k}$$

$$Ra = Gr\ \mathrm{Pr}$$

$$Gr = \frac{\rho\,\beta \left(T_w - T_\infty\right) L^3}{\left(\mu / \rho\right)^2}$$

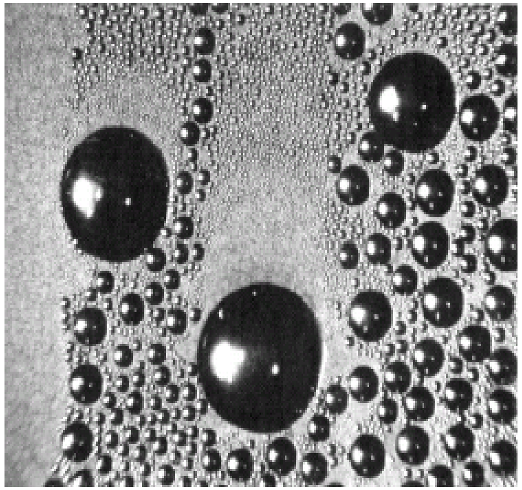
Boundary conditions: review

Boiling



Boundary conditions: condensation

$$\overline{\mathrm{Nu}}_D = 0.64\left\{\frac{\rho_f u_\infty D}{\mu_f}\left[1+\left(1+1.69\,\frac{g h'_{fg}\mu_f D}{u_\infty^2 k_f (T_{\mathrm{sat}}-T_w)}\right)^{1/2}\right]\right\}^{1/2}$$



b. Typical photograph of dropwise condensation provided by Professor Borivoje B. Mikić. Notice the dry paths on the left and in the wake of the middle droplet.

Time integration

$$\rho Cp \frac{\partial T}{\partial t} - \nabla \cdot (k \nabla T) = Q \qquad x \in \Omega$$

$$T = T_{imp} \qquad x \in \Gamma_T$$

$$q_n = q_{n_{imp}} \qquad x \in \Gamma_T$$

$$T(x,0) = T_0 \qquad \textit{initial} \quad \textit{condition}$$

$$\tilde{T}(x,t) = \underline{H}(x) \cdot \underline{\hat{T}}(t) \qquad \underline{x} = \underline{H} \cdot \underline{\hat{x}}$$

$$\int_\Omega \underline{H}^T \rho Cp \frac{\partial \tilde{T}}{\partial t} d\Omega - \int_\Omega \underline{H}^T \nabla \cdot (k \nabla \tilde{T}) d\Omega =$$

$$\int_\Omega \underline{H}^T Q \, d\Omega + \int_{\Gamma_q} \underline{H}^T (q_n - q_{n_{imp}}) d\Gamma$$

Part integration

Time integration

$$\underline{\underline{M}} \cdot \underline{\dot{\hat{T}}} + \underline{\underline{K}} \cdot \underline{\hat{T}} = \underline{F}$$

$$M_{ij}^{G} = \sum_{e} \int_{\Omega^e} h_i \, \rho C_p \, h_j \, d\Omega$$

$$K_{ij}^{G} = \sum_{e} \int_{\Omega^e} B_{im} \, k_{mp} \, B_{pj} \, d\Omega$$

$$\begin{bmatrix} \frac{\partial \tilde{T}}{\partial x} \\ \frac{\partial \tilde{T}}{\partial y} \\ \frac{\partial \tilde{T}}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_2}{\partial x} & \cdots & \frac{\partial h_n}{\partial x} \\ \frac{\partial h_1}{\partial y} & \frac{\partial h_2}{\partial y} & \cdots & \frac{\partial h_n}{\partial y} \\ \frac{\partial h_1}{\partial z} & \frac{\partial h_2}{\partial z} & \cdots & \frac{\partial h_n}{\partial z} \end{bmatrix} \cdot \underline{\hat{T}} = \underline{B} \cdot \underline{\hat{T}}$$

$$\underline{\underline{k}} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}$$

$$F_i^G = \sum_{e} \int_{\Omega} h_i \, Q \, d\Omega - \sum_{e} \int_{\Gamma_q} h_i \, q_{n_{imp}}$$

Time integration

Time integration with convection therms

$$\rho C_p \frac{\partial T}{\partial t} + \rho C_p \underline{v} \cdot \underline{\nabla} T - \underline{\nabla} \cdot \left(\underline{\underline{k}} \cdot \underline{\nabla} T \right) = Q$$

$$\underline{\underline{M}} \cdot \dot{\underline{\hat{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}} \right) \cdot \underline{\hat{T}} = \underline{F}$$

$$\underline{\underline{\mathbf{M}}} = \sum_{\epsilon = 1}^{NE} \left(\underline{\underline{\mathbf{M}}}^{G^{(\epsilon)}} + \underline{\underline{\mathbf{M}}}^{P^{(\epsilon)}} \right) \quad ; \quad \underline{\underline{\mathbf{N}}} = \sum_{\epsilon = 1}^{NE} \left(\underline{\underline{\mathbf{N}}}^{G^{(\epsilon)}} + \underline{\underline{\mathbf{N}}}^{P^{(\epsilon)}} \right)$$

$$\underline{\underline{\mathbf{K}}} = \sum_{\epsilon = 1}^{NE} \left(\underline{\underline{\mathbf{K}}}^{G^{(\epsilon)}} + \underline{\underline{\mathbf{K}}}^{P^{(\epsilon)}} \right) \quad ; \quad \underline{\mathbf{F}} = \sum_{\epsilon = 1}^{NE} \left(\underline{\mathbf{F}}^{G^{(\epsilon)}} + \underline{\mathbf{F}}^{P^{(\epsilon)}} \right)$$

Time integration

Time integration with convection therms

$$M_{ij}^{G(e)} = \int_{\Omega^e} h_i \, \rho C_p h_j \, d\Omega$$

$$K_{ij}^{Ge} = \int_{\Omega^e} B_{im} \, k_{mp} \, B_{pj} \, d\Omega \quad ; \quad N_{ij}^{Ge} = \int_{\Omega^e} h_i \, v_p \, B_{pj} \, d\Omega$$

$$F_i^{Ge} = \int_{\Omega^e} h_i \, q_v \, d\Omega - \int_{\Gamma_{q^e}} h_i \, q_{n_{imp}}$$

Time integration

Time integration with convection therms

$$M_{ij}^{P^{(e)}} = \int_{\Omega^e} W_i \; \rho C_p h_j \; d\Omega$$

$$K_{ij}^{P^{(e)}} = - \int_{\Omega^e} W_i \; \underline{\nabla} \cdot \left(\underline{\underline{\mathbf{k}}} \cdot \underline{\nabla} h_j \right) \; d\Omega$$

$$N_{ij}^{P^{(e)}} = \int_{\Omega^e} W_i \; \rho C_p \underline{\mathbf{v}} \cdot \underline{\nabla} h_j \; d\Omega$$

$$F_i^{P^{(e)}} = \int_{\Omega^e} W_i \; q_v \; d\Omega$$

Time integration

Alpha Method

$$\underline{\underline{M}} \cdot \dot{\hat{\underline{T}}} + \left(\underline{\underline{N}} + \underline{\underline{K}}\right) \cdot \hat{\underline{T}} = \underline{F}$$

The objective is to obtain an approximation for ${}^{t+\Delta t}T$ given the value of ${}^{t}T$ ${}^{t}F$ and ${}^{t+\Delta t}F$

Alpha Method seeks to satisfy the differential equation in

$$t + \alpha \Delta t \quad ; \quad 0 \leq \alpha \leq 1$$

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Time integration: Alpha Method

$$\underline{\underline{M}}\cdot\dot{\hat{\underline{T}}}+\left(\underline{\underline{N}}+\underline{\underline{K}}\right)\cdot\hat{\underline{T}}=\underline{F}$$

$$^{t+\alpha\Delta t}\dot{\hat{\underline{T}}}=\frac{^{t+\Delta t}\hat{\underline{T}}-{}^{t}\hat{\underline{T}}}{\Delta t}$$

$$^{t+\alpha\Delta t}\hat{\underline{T}}={}^{t}\hat{\underline{T}}+\frac{^{t+\Delta t}\hat{\underline{T}}-{}^{t}\hat{\underline{T}}}{\Delta t}\alpha\,\Delta t+\vartheta\left(\Delta t^{2}\right)=\left(1-\alpha\right){}^{t}\hat{\underline{T}}+\alpha\,{}^{t+\Delta t}\hat{\underline{T}}$$

$$^{t+\alpha\Delta t}\underline{F}=\left(1-\alpha\right){}^{t}\underline{F}+\alpha\,{}^{t+\Delta t}\underline{F}$$

Time integration: Alpha Method

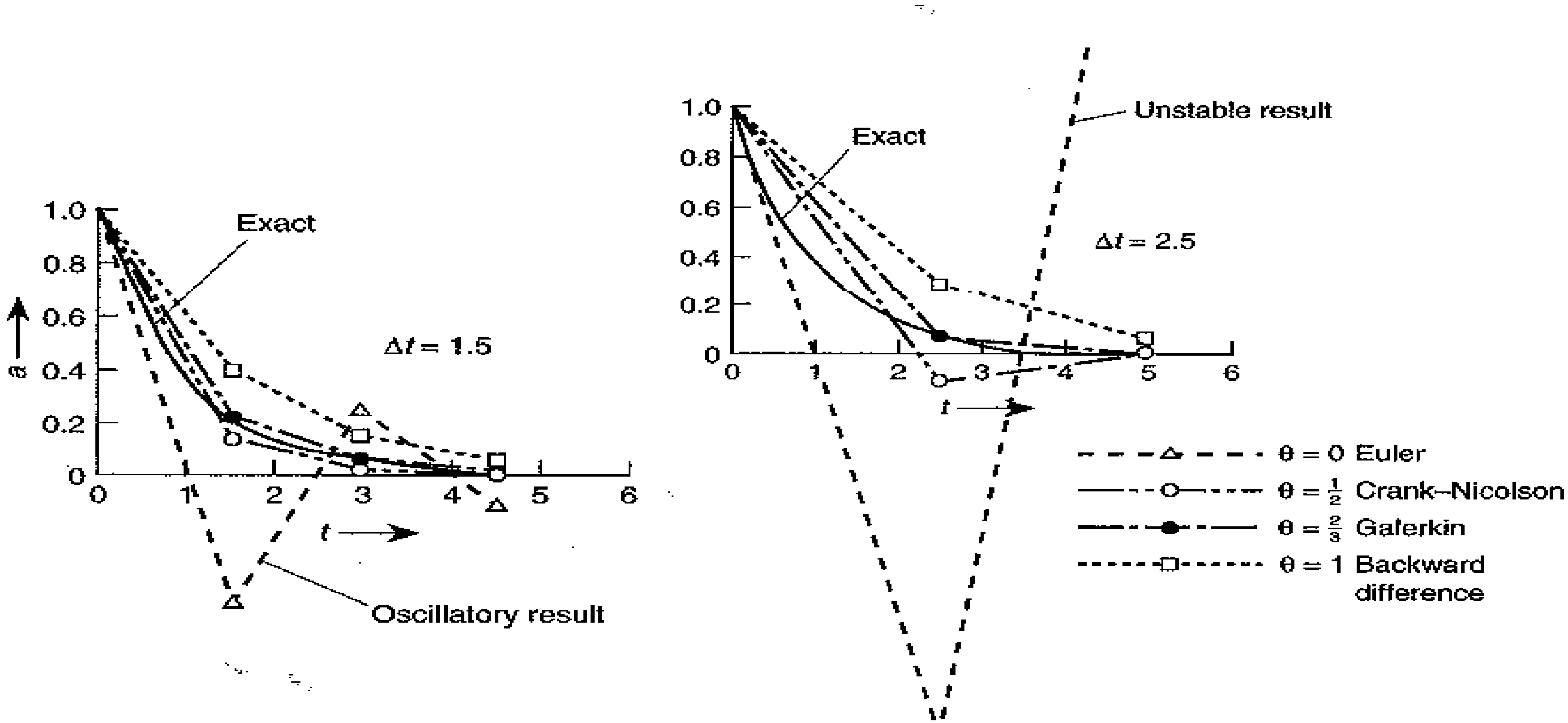
$$\left[\underline{\underline{\mathbf{M}}} + \alpha\,\Delta t\left(\underline{\underline{\mathbf{N}}} + \underline{\underline{\mathbf{K}}}\right)\right]\cdot\,{}^{t+\Delta t}\underline{\widehat{\mathbf{T}}} = \alpha\,\Delta t\,{}^{t+\Delta t}\underline{\mathbf{F}} + (1-\alpha)\,\Delta t\,{}^{t}\underline{\mathbf{F}}$$
$$-\,(1-\alpha)\,\Delta t\left(\underline{\underline{\mathbf{N}}} + \underline{\underline{\mathbf{K}}}\right)\cdot\,{}^{t}\underline{\widehat{\mathbf{T}}} + \underline{\underline{\mathbf{M}}}\cdot\,{}^{t}\underline{\widehat{\mathbf{T}}}$$

$\alpha = 1$ Implicit Euler backward Method, unconditionally stable $\vartheta(\Delta t)$

$\alpha = 0$ Explicit Euler forward Method, conditionally stable $\vartheta(\Delta t)$

$\alpha = \frac{1}{2}$ Implicit trapezoidal rule, unconditionally stable $\vartheta(\Delta t^2)$
Cranck Nicolson method

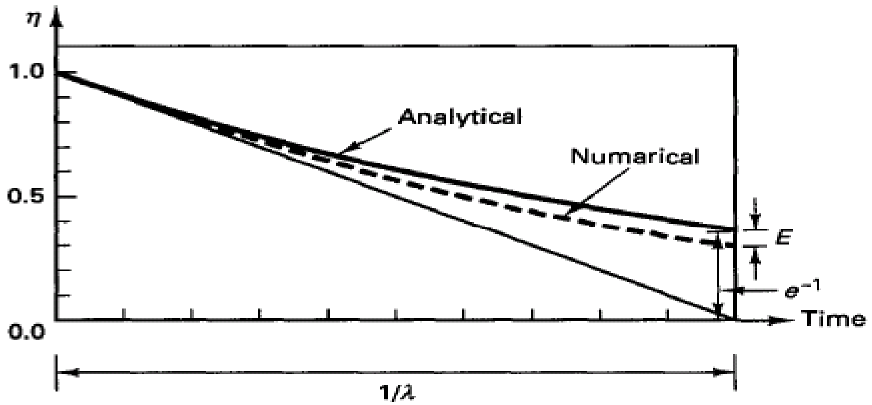
Time integration: Alpha Method

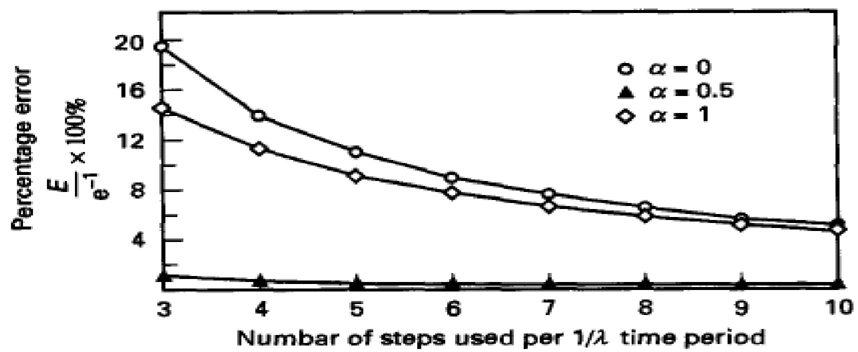


From Zienkiewicz & Taylor, The Finite Element Method

Time integration: Alpha Method

Approximation error





Time integration

Penetration depth measures the distance or thickness of thermal energy propagating into the surface through conduction.

$$\gamma = 4\sqrt{\frac{k}{\rho C_p} t}$$

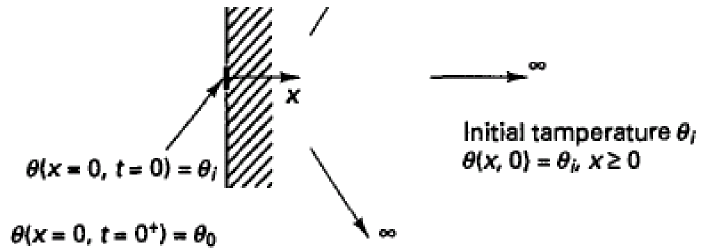
$$\frac{T(\gamma) - T_{initial}}{T_{BC} - T_{initial}} < 0.01$$

t_{min} is the minimum time at which temperature results are desired.

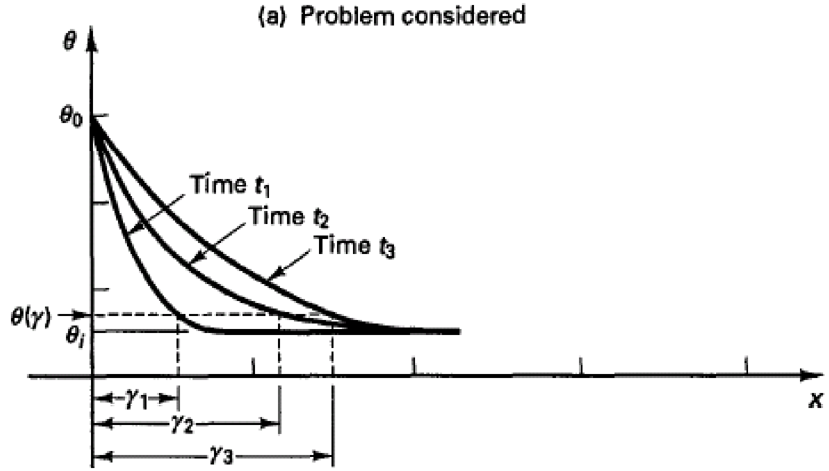
N is the elements number to discriteze the penetration depth

$$\Delta x = \frac{4}{N}\sqrt{\frac{k}{\rho C_p} t_{min}}$$

Tipically N = 6 to 10



(a) Problem considered



(b) Penetration depths at three different times (schematic presentation)

$T = \theta$ *From Bathe, Finite Element Procedures*

Non-linear equations

Steady State

$$\underline{F}\left(\underline{\hat{T}}\right) = \underline{R} \quad \Longrightarrow \quad \underline{\underline{K}}\left(T\right) \cdot \underline{\hat{T}} = \underline{R}$$

Transient State

$${}^{t+\Delta t}\underline{F}\left({}^{t+\Delta t}\underline{\hat{T}}, {}^{t}\underline{\hat{T}} \right) = {}^{t+\Delta t}\underline{R}\left({}^{t}\underline{\hat{T}} \right)$$

$$\Downarrow$$

$$\underline{\underline{K}}\left({}^{t+\Delta t}T, {}^{t}T \right) \cdot {}^{t+\Delta t}\underline{\hat{T}} = {}^{t+\Delta t}\underline{R}\left({}^{t}T \right)$$

Non-linear equations: Picard Method

It is called successive substitutions method.

Starting with an initial guess

Evaluate

$$k = k + 1$$

$$\underline{\underline{K}}\left({}^{t+\Delta t}\underline{\hat{T}}^{(k-1)}, {}^{t}T \right) \cdot {}^{t+\Delta t}\underline{\hat{T}}^{(k)} = {}^{t+\Delta t}\underline{R}\left({}^{t}T \right)$$

Until the result no longer changes to within a specified tolerance

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Non-linear equations: Picard Method

Picard's method is the easiest method to program and usually has large areas of convergence .

Converges linearly and for many problems its convergence rate is very smooth

The most important application of Picard's method is to use it as the first iterations of the Newton-Raphson method .

Non-linear equations: Newton-Raphson Method

Historical Note.

Newton's work was done in 1669 but published much later. Numerical methods related to the Newton Method were used by al-Kash, Viete, Briggs, and Oughtred, all many years before Newton.

Raphson, some 20 years after Newton, got close to Newton Equation, but only for *polynomials* of degree 3, 4, 5, . . . , 10.

Raphson, like Newton, seems unaware of the connection between his method and the derivative. The connection was made about 50 years later (Simpson, Euler), and the Newton Method finally moved beyond polynomial equations. The familiar geometric interpretation of the Newton Method may have been first used by Mourraille (1768). Analysis of the convergence of the Newton Method had to wait until Fourier and Cauchy in the 1820s.

Non-linear equations: Newton-Raphson Method

Steady state problem $\underline{R} - \underline{F}\left(\underline{\hat{T}}\right) = \underline{0} \quad ; \quad \underline{F}\left(\underline{\hat{T}}\right) = \underline{\underline{K}}\left(\underline{\hat{T}}\right) \cdot \underline{\hat{T}}$

Linearized $$\underline{F} = \underline{F}^{(k-1)} + \left.\frac{\partial \underline{F}}{\partial \hat{T}_j}\right|^{(k-1)} \Delta \hat{T}_j^{(k)} \quad ; \quad j=1,NEQ$$

$$\Delta \hat{T}_j^{(k)} = \hat{T}_j^{(k)} - \hat{T}_j^{(k-1)}$$

Call tangent matrix $$K_{T_{ij}} = \left.\frac{\partial F_i}{\partial \hat{T}_j}\right|$$

Non-linear equations: Newton-Raphson Method

$$\underline{R} - \underline{F}^{(k-1)} - \underline{\underline{K}}_T^{(k-1)} \cdot \Delta\hat{\underline{T}}^{(k)} = \underline{0}$$

$$\underline{\underline{K}}_T^{(k-1)} \cdot \Delta\hat{\underline{T}}^{(k)} = \underline{R} - \underline{F}^{(k-1)}$$

$$\Delta\hat{\underline{T}}^{(k)} = \hat{\underline{T}}^{(k)} - \hat{\underline{T}}^{(k-1)}$$

Start conditions $\hat{\underline{T}}^{(0)} = \hat{\underline{T}}_{data}$

Non-linear equations: Newton-Raphson Method

Transient state problem

$$ {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F} = \underline{0} \qquad ; \qquad {}^{t+\Delta t}\underline{F} = {}^{t+\Delta t}\underline{\underline{K}}\left({}^{t+\Delta t}\underline{\hat{T}}\right) \cdot {}^{t+\Delta t}\underline{\hat{T}} $$

Linearized

$$ {}^{t+\Delta t}\underline{F} = {}^{t+\Delta t}\underline{F}^{(k-1)} + \left.\frac{\partial\, {}^{t+\Delta t}\underline{F}}{\partial\, {}^{t+\Delta t}\hat{T}_j}\right|^{(k-1)} {}^{t+\Delta t}\Delta\hat{T}_j^{(k)} \qquad ; \qquad j=1,NEQ $$

$$ {}^{t+\Delta t}\Delta\hat{T}_j^{(k)} = {}^{t+\Delta t}\hat{T}_j^{(k)} - {}^{t+\Delta t}\hat{T}_j^{(k-1)} $$

Call tangent matrix

$$ {}^{t+\Delta t}K_{T_{ij}} = \left.\frac{\partial\, {}^{t+\Delta t}F_i}{\partial\, {}^{t+\Delta t}\hat{T}_j}\right| $$

Non-linear equations: Newton-Raphson Method

$$^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)} - {}^{t+\Delta t}\underline{\underline{K}}_T^{(k-1)} \cdot {}^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)} = \underline{0}$$

$$^{t+\Delta t}\underline{\underline{K}}_T^{(k-1)} \cdot {}^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)} = {}^{t+\Delta t}\underline{R} - {}^{t+\Delta t}\underline{F}^{(k-1)}$$

$$^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)} = {}^{t+\Delta t}\underline{\hat{T}}^{(k)} - {}^{t+\Delta t}\underline{\hat{T}}^{(k-1)}$$

Start conditions

$$^{t+\Delta t}\underline{\hat{T}}^{(0)} = {}^{t}\underline{\hat{T}} \quad ; \quad {}^{t+\Delta t}\underline{\underline{K}}_T^{(0)} = {}^{t}\underline{\underline{K}}_T \quad ; \quad {}^{t+\Delta t}\underline{F}^{(0)} = {}^{t}\underline{F}$$

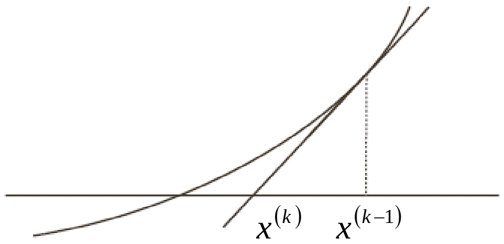
Non-linear equations: Newton-Raphson Method

For one degree of freedom

$$f_{(x)} = 0$$

$$f_{(x^{(k-1)})} + f'_{(x^{(k-1)})}\left(x^{(k)} - x^{(k-1)}\right) = 0$$

$$x^{(k)} = x^{(k-1)} - \frac{f_{(x^{(k-1)})}}{f'_{(x^{(k-1)})}}$$



Non-linear equations: Newton-Raphson Method

Example: We use the Newton-Raphson Method to find a non-zero solution of

$$x = 2\sin x$$

(a) Start $x^{(0)} = 1.1$

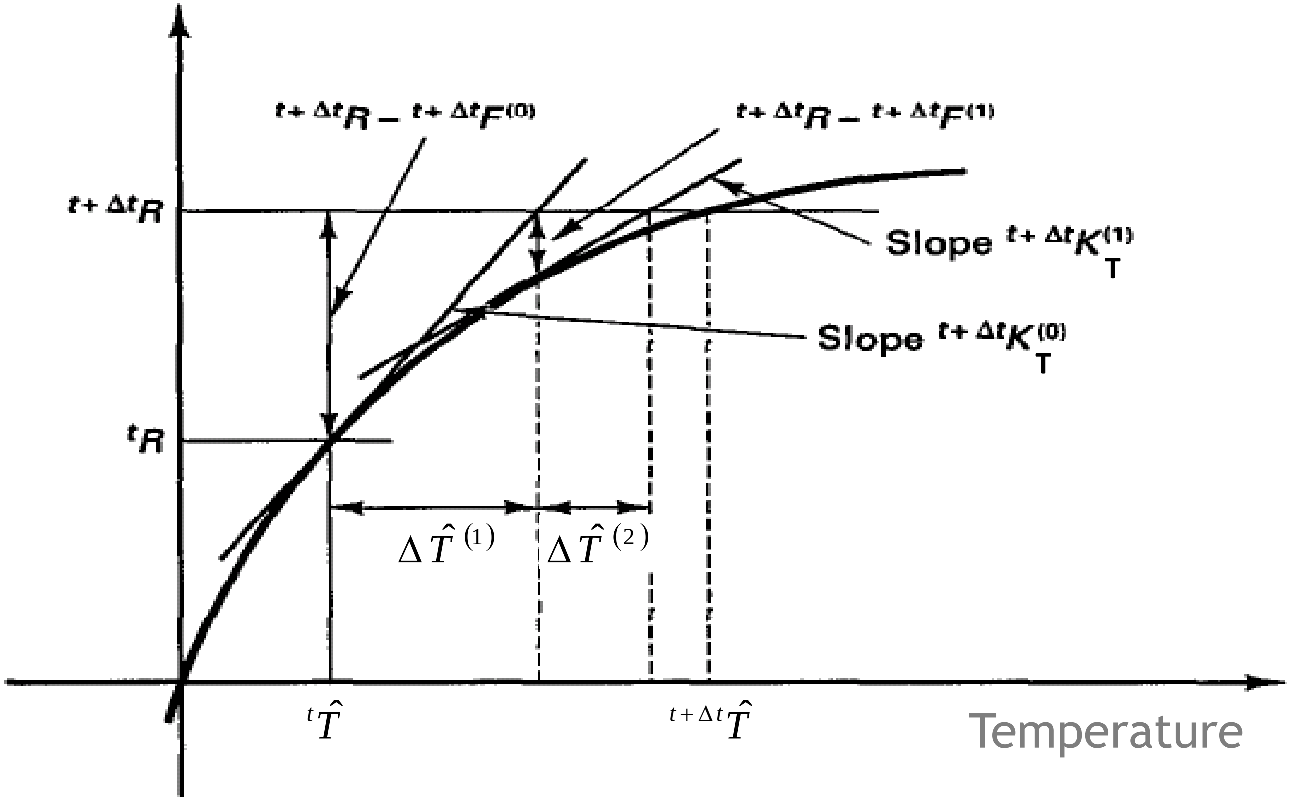
(b) Start $x^{(0)} = 1.5$

Non-linear equations: Newton-Raphson Method

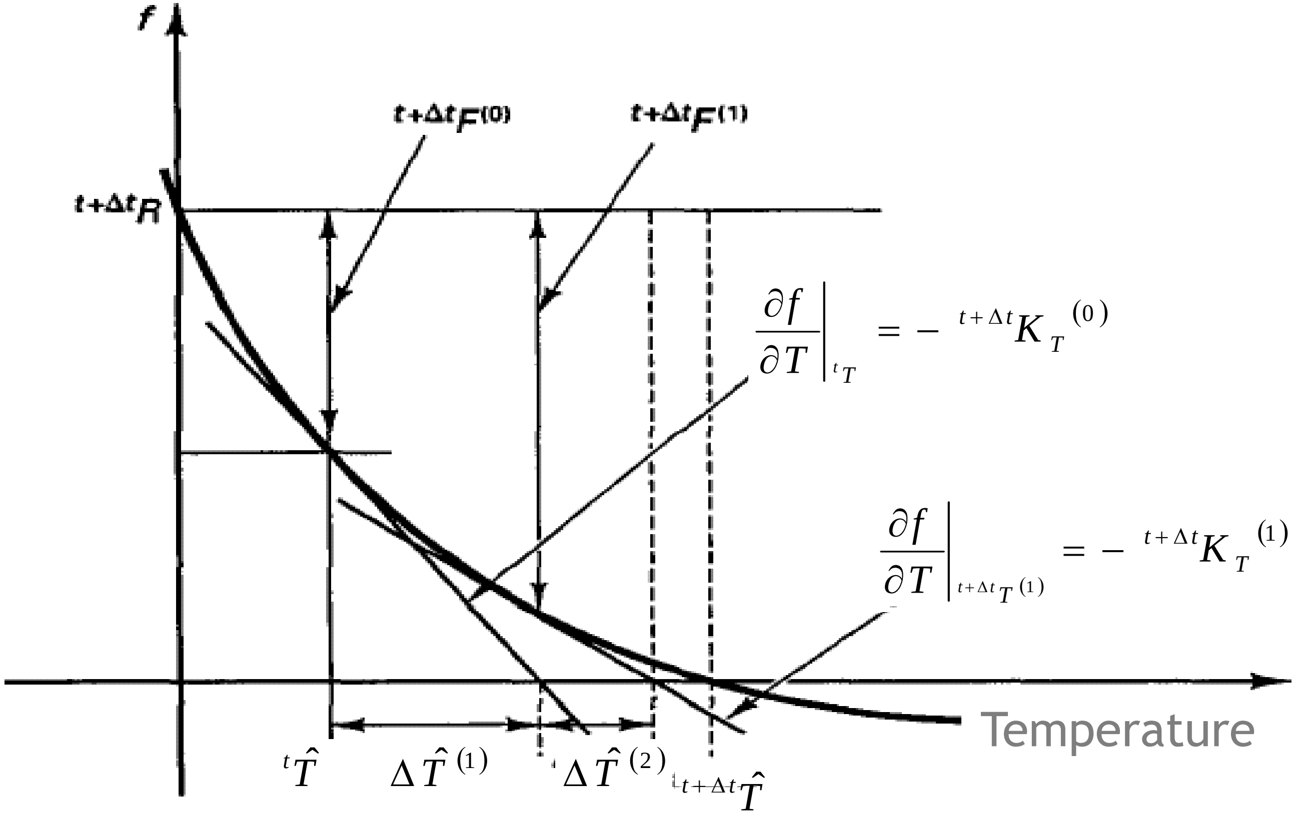
If the initial estimate is not close enough to the root, the Newton-Raphson Method may not converge, or may converge to the wrong root.

The successive estimates of the Newton-Raphson Method may converge to the root too slowly, or may not converge at all.

Non-linear equations: Newton-Raphson Method



Non-linear equations: Newton-Raphson Method



Non-linear equations: Newton-Raphson Method

Convergence

$$
\begin{aligned}
f_{(x)} &= 0 \\
f_{(x^{(k-1)})} + f'_{(x^{(k-1)})}\left(x^{(k)} - x^{(k-1)}\right) &= 0 \\
x^{(k)} &= x^{(k-1)} - \frac{f_{(x^{(k-1)})}}{f'_{(x^{(k-1)})}}
\end{aligned}
$$

Quadratic convergence when converges

Non-linear equations: Newton-Raphson Method

Convergence

(1) First property

- If the tangent matrix ${}^{t+\Delta t}\underline{\underline{K}}_T^{(k-1)}$ is nonsingular
- If ${}^{t+\Delta t}\underline{F}^{(k-1)}$ and its first derivatives with respect to ${}^{t+\Delta t}\underline{\hat{T}}^{(k-1)}$ are continuous in a neighborhood of the solution ${}^{t+\Delta t}\underline{\hat{T}}^{*}$
- If ${}^{t+\Delta t}\underline{\hat{T}}^{(k-1)}$ will be closer to ${}^{t+\Delta t}\underline{\hat{T}}^{*}$

than ${}^{t+\Delta t}\underline{\hat{T}}^{(k)}$ and the sequence of iterative solutions converges to ${}^{t+\Delta t}\underline{\hat{T}}^{*}$

Non-linear equations: Newton-Raphson Method

Convergence

(2) Second property – Lipschitz continuity

- If the tangent matrix satisfies $\left\| {}^{t+\Delta t}\underline{\underline{K}}_T^{(k)} - {}^{t+\Delta t}\underline{\underline{K}}_T^{(k-1)} \right\| \leq L \left\| {}^{t+\Delta t}\underline{\hat{T}}^{(k)} - {}^{t+\Delta t}\underline{\hat{T}}^{(k-1)} \right\|$

  for all ${}^{t+\Delta t}\underline{\hat{T}}^{(k)}$ *and* ${}^{t+\Delta t}\underline{\hat{T}}^{(k-1)}$ in the neighborhood of ${}^{t+\Delta t}\underline{\hat{T}}^{*}$

  and $L>0$

then convergence is quadratic.

This means that if the error after iteration *(k)* is the order e, then the error after iteration *(k+1)* will be of the order e^2

Modified Newton-Raphson Method

N-R iteration is recognized as an expensive computational cost per iteration due to the calculation and factorization of the tangent matrix. Then, the use of a modification of the full N-R algorithm can be effective.

Maintains the $f'(x)$ o tangent matrix $^{t+\Delta t}\underline{\underline{K}}_T^{(k-1)}$ constant during the iterations or it is modified each n iterations

Advantage: saving computational effort

Disadvantage: loss of quadratic convergence

The choice of time step depend on the degree of non-linearities

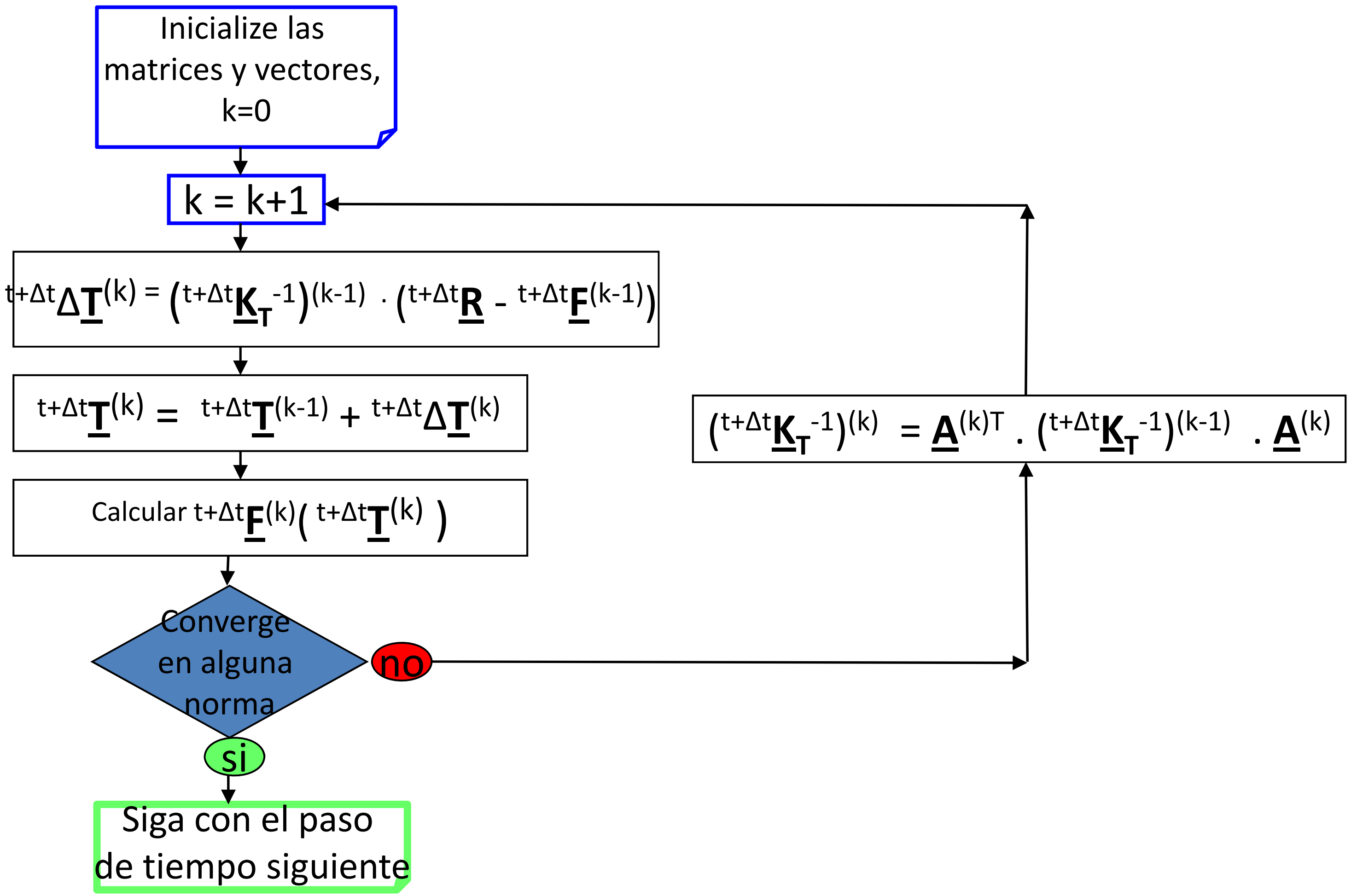
Quasi-Newton Methods: BFGS

As an alternative to forms of N-R iteration, a class of methods known as matrix update methods or quasi-Newton methods has been developed .

These methods involve updating the coefficient matrix to provide a secant approximation to the matrix from iteration *(k-1)* to *(k)*.

BFGS: Broyden, Fletcher, Goldfarb and Shanno method

Quasi-Newton Methods: BFGS



Quasi-Newton Methods: BFGS

$$\left({}^{t+\Delta t}\underline{\underline{\mathbf{K}}}_T^{-1}\right)^{(k)} = \underline{\underline{\mathrm{A}}}^{(k)T} \cdot \left({}^{t+\Delta t}\underline{\underline{\mathbf{K}}}_T^{-1}\right)^{(k-1)} \cdot \underline{\underline{\mathrm{A}}}^{(k)}$$

$$\underline{\underline{\mathbf{A}}}^{(k)} = \underline{\underline{\mathbf{I}}} + \underline{\mathbf{v}}^{(k)}\ \underline{\mathbf{w}}^{(k)T}$$

$$\underline{\mathbf{v}}^{(k)} = -\,c^{(k)}\ {}^{t+\Delta t}\underline{\underline{\mathbf{K}}}^{(k-1)} \cdot {}^{t+\Delta t}\Delta\underline{\mathbf{T}}^{(k)} - {}^{t+\Delta t}\Delta\underline{\mathbf{F}}^{(k)}$$

$$c^{(k)} = \left[\frac{{}^{t+\Delta t}\Delta\underline{\mathbf{T}}^{(k)T} \cdot {}^{t+\Delta t}\Delta\underline{\mathbf{F}}^{(k)}}{{}^{t+\Delta t}\Delta\underline{\mathbf{T}}^{(k)T} \cdot {}^{t+\Delta t}\underline{\underline{\mathbf{K}}}_T^{(k-1)} \cdot {}^{t+\Delta t}\Delta\underline{\mathbf{T}}^{(k)}}\right]^{\frac{1}{2}}$$

$$\underline{\mathbf{w}}^{(k)} = \frac{{}^{t+\Delta t}\Delta\underline{\mathbf{T}}^{(k)}}{{}^{t+\Delta t}\Delta\underline{\mathbf{T}}^{(k)T} \cdot {}^{t+\Delta t}\Delta\underline{\mathbf{F}}^{(k)}}$$

Quasi-Newton Methods: BFGS

$$ {}^{t+\Delta t}\Delta \underline{\hat{T}}^{(k)} = {}^{t+\Delta t}\underline{\hat{T}}^{(k)} - {}^{t+\Delta t}\underline{\hat{T}}^{(k-1)} $$

$$ {}^{t+\Delta t}\Delta \underline{F}^{(k)} = {}^{t+\Delta t}\underline{F}^{(k)} - {}^{t+\Delta t}\underline{F}^{(k-1)} $$

Since the product $\left({}^{t+\Delta t}\underline{\underline{\mathbf{K}}}_T^{-1}\right)^{(k)} = \underline{\underline{\mathrm{A}}}^{(k)T} \cdot \left({}^{t+\Delta t}\underline{\underline{\mathbf{K}}}_T^{-1}\right)^{(k-1)} \cdot \underline{\underline{\mathrm{A}}}^{(k)}$

is positive definite and symmetric, to avoid numerically problems, the condition number is calculated.

The update is performed if: $c^{(k)} < n \quad \left(as\ example\ n = 10^5\right)$

BFGS with linear searches

$$^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)} = {}^{t+\Delta t}\underline{\hat{T}}^{(k)} - \beta \; {}^{t+\Delta t}\underline{\hat{T}}^{(k-1)}$$

b is a scalar multiplier

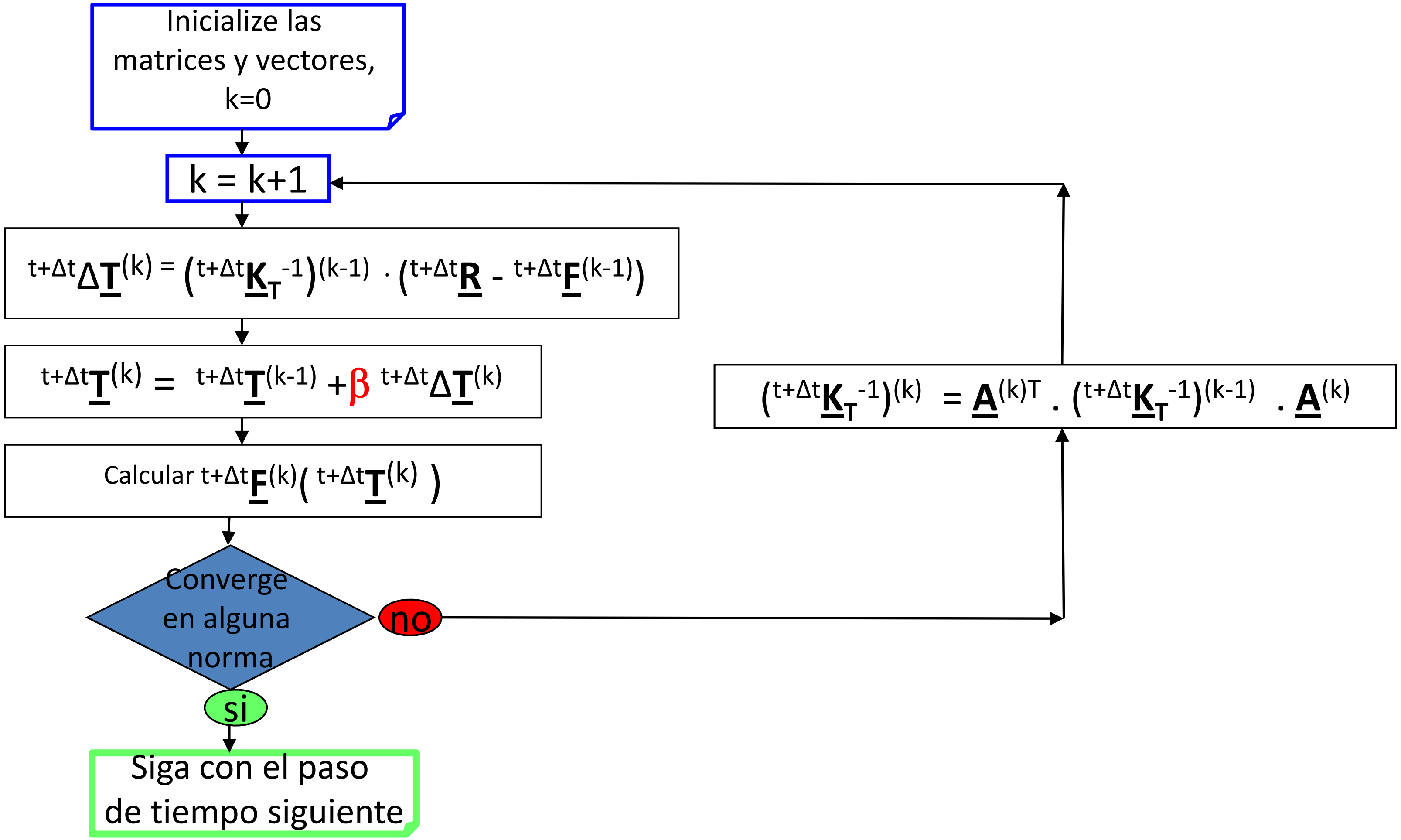
It is varied until the component of the out-of-balance loads in the direction $^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)}$ is small.

$$^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)T}\left({}^{t+\Delta t}\Delta\underline{R} - {}^{t+\Delta t}\Delta\underline{F}^{(k)}\right) \leq TOL \; {}^{t+\Delta t}\Delta\underline{\hat{T}}^{(k)T}\left({}^{t+\Delta t}\Delta\underline{R} - {}^{t+\Delta t}\Delta\underline{F}^{(k-1)}\right)$$

Linear searches are made with simple algorithms such as bisection

Linear searches are computationally expensive because they must calculate multiple times in each iteration $^{t+\Delta t}\underline{F}^{(k)}$

BFGS with linear searches



Convergence criteria

1) Convergence in temperatures $\left\| {}^{t+\Delta t}\Delta \hat{\underline{T}}^{(k)} \right\| < DTOL$

2) Convergence in porcentual values $\dfrac{\left\| {}^{t+\Delta t}\Delta \hat{\underline{T}}^{(k)} \right\|}{\left\| {}^{t+\Delta t}\Delta \hat{\underline{T}}^{(k-1)} \right\|} < ETOL$

$$\|\underline{a}\|_2 = \sqrt{\sum_{i=1}^{n} \left(a^i\right)^2} \quad ; \quad \|\underline{a}\|_1 = \sum_{i=1}^{n} \left|a^i\right| \quad ; \quad \|\underline{a}\|_\infty = \max \left|a^i\right|$$

Examples on transitory heat transfer problems

Exercise 1: Obtain the FEA formulation for the Linear Transitory heat transfer problem considering convection. Analyze the stability of the different time integration

Exercise 2: Consider the transitory heat transfer problem in a 1D beam discretized with 10 regular elements. Solve the finite element model with time integration for different alpha values (0; 0.5 and 1) for the following cases:

a) $\tau_{final} = 0.05 \quad \delta\tau = 0.05$
b) $\tau_{final} = 0.5 \quad \delta\tau = 0.05$
c) $\tau_{final} = 0.00005 \quad \delta\tau = 0.00005$
d) $\tau_{final} = 0.5 \quad \delta\tau = 0.00005$

Heat transfer equation $\quad \frac{\partial T}{\partial t} = \eta \frac{\partial^2 T}{\partial x^2} \qquad 0 \leq x \leq L \quad \wedge \quad t > 0$

Border Condition $\quad T_{(L,t)} = T_L \quad \wedge \quad \eta \left.\frac{\partial T}{\partial x}\right|_{(0,t)} = 0 \qquad t \geq 0$

Initial Condition $\quad T_{(x,0)} = 0 \qquad 0 \leq x < L$

Use this non-dimensional numbers for the analysis: $y = \frac{x}{L} \quad ; \quad \theta = 1 - \frac{T}{T_L} \quad ; \quad \tau = \frac{t\eta}{L^2}$

Examples on transitory heat transfer problems

Exercise 3: Consider a 90º semi-infinite cylinder. Sides AB and BC are subjected to prescribed temperature of 50º. The initial temperature profile is 0º. The heat capacity of the material is constant. Perform a transient analysis to calculate the temperature distribution within the semi-infinite domain at different values of time. Use the Euler Backward, Cranck Nicholson and Euler Forward Method.

The domain is discretized using a 10 × 10 mesh of 4-node 2-D conduction elements.
The conduction matrix is evaluated using a consistent heat capacity matrix. The time step is $\Delta t = 0.016$.

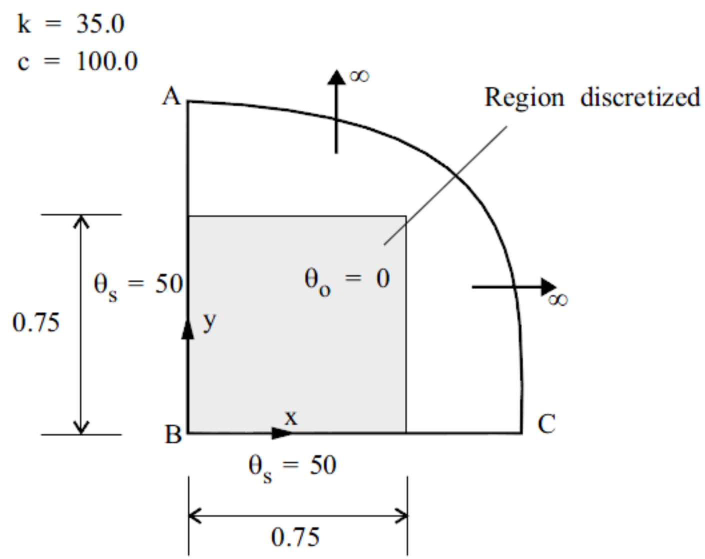


Figure T.8