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Simulación y Tecnología

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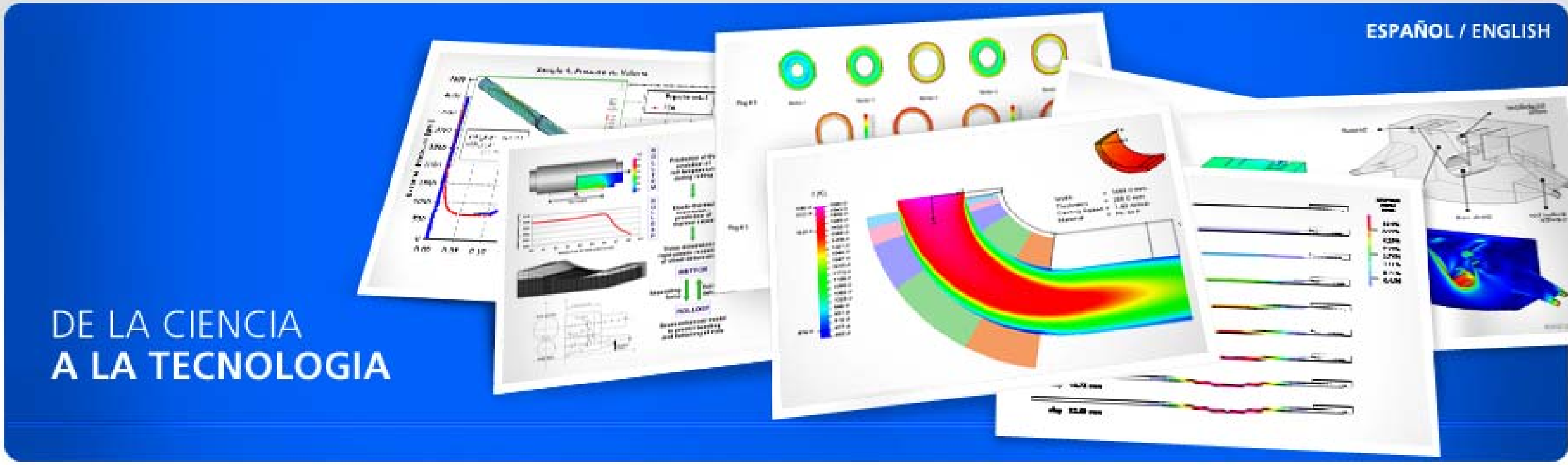

FINITE ELEMENT METHOD IN FLUID DYNAMICS

Part 4: Turbulent flow

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Turbulent Movement

The main characteristics of the turbulent flow is the random fluctuations in the variables (fluid velocity, pressure, etc.) so that statistically distinct averages can be discerned.

The fluctuations are due to a continuous generation and movement of eddies in the flow.

It is worth pointing out that a turbulent flow is described by the usual continuum mechanics equations because the smallest length scales in a turbulent flow are much higher than the molecular length scales.

The mathematical description of a turbulent flow using the time averaged Navier- Stokes equations leads to the development of turbulence models to close the problem.

These turbulence models do not simulate the details of the turbulent motion but only the effect of turbulence on the mean-flow behavior.

Turbulent Movement

Large portions of fluid (eddies) have a random variation of the velocity components in three dimensions superimposes on a mean motion.

The eddies can change shape, stretch, rotate, or break into two or more eddies. Its size determines the scale of turbulence.

At a given time the flow has a size distribution of eddies.

The largest eddies are the size of the flow (depth of a stair) while the small eddies can be of the order of one millimeter (sufficiently larger than the size of the molecules).

Small eddies (smaller scale length) occur in small time scale and are considered statistically independent of large scale turbulent flow and mean flow.

The processes involving turbulent flows that change the length scale of the eddies.

Turbulent Movement

The large eddies that are continually forming eddies break up into smaller and smaller and smaller until they finally dissipate as shear flow

Turbulent flows are dissipative. The small scale is reached when the eddies lose energy by the action of viscous forces and convert internal energy on the small eddies.

The turbulent fluxes are diffusive. The diffusivity of turbulence causes rapid mixing and increases the speed of transfer of momentum, heat and mass.

Turbulence is rotational, three-dimensional and is characterized by high levels of fluctuating vorticity.

The intensity of fluctuations is variable, but not strictly periodic and random distribution for a wide range or spectrum of frequencies.

Turbulent Movement

The small-scale turbulent fluctuations are independent of mean flow. This is natural because it does not consider the movement of these small eddies as a whole, but the relative motion of particles.

Small scales are independent of flow parameters and therefore can only depend on ε and μ, but not on larger scales. The combinations of these quantities are:

Length scale $\eta = \left(\frac{\upsilon^3}{\varepsilon}\right)^{\frac{1}{4}}$

Time scale $\tau = \left(\frac{\upsilon}{\varepsilon}\right)^{\frac{1}{2}}$

Velocity scale $\upsilon = \left(\frac{\mu}{\rho}\varepsilon\right)^{\frac{1}{4}}$

Turbulent Movement

The Reynolds Number is $Re = \eta v \rho / \mu = 1$,

Kolmogorov microscales

The ratio of the scales involved in the problem of turbulence is:

- $$\frac{\eta}{l} \sim \left(\frac{vl\rho}{\mu}\right)^{-\frac{3}{4}} = \mathrm{Re}^{-\frac{3}{4}}$$
- $$\frac{t}{\tau} \sim \left(\frac{vl\rho}{\mu}\right)^{-\frac{1}{2}} = \mathrm{Re}^{-\frac{1}{2}}$$

Turbulent Movement

To solve the problem taking into account all scales between the external and internal (exactly solving the Navier Stokes) are needed in three dimensions on the order of $Re^{9/4}$ degrees of freedom.

For a Reynolds number of 10000 (the reasonable minimum in an industrial problem) then it would take 10^9 degrees of freedom, 8 Gbytes to store only the unknowns, it is not possible with current computer capabilities.

Turbulent Movement

Continuity equation:

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_i}{\partial x_i} = 0 \quad \Rightarrow \quad \partial_i u_i = 0$$

Navier-Stokes equations

$$\rho \frac{D u_i}{D t} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i \qquad \sigma_{ij} = -p\,\delta_{ij} + 2\mu S'_{ij} \qquad S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

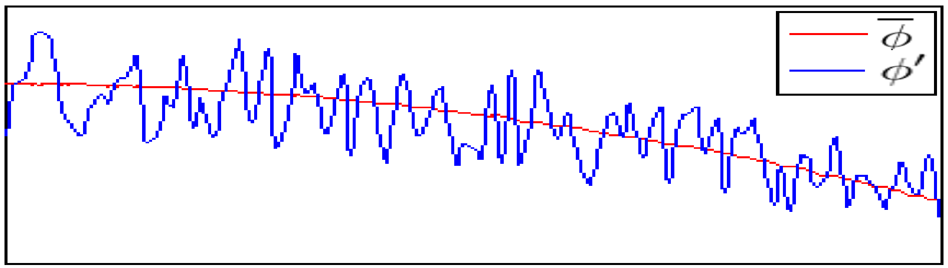
$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \partial_j u_i = -\partial_i p + \partial_j \left[\mu\left(\partial_j u_i + \partial_i u_j\right)\right] + \rho f_i$$

Turbulent Models

"A turbulence model is defined as a set of equations (algebraic and differential) which determines the turbulence transport terms and thus closes the system of the equations to be modelled. The turbulence does not simulate the details of the turbulent movement but the effect of turbulence on the behavior the medium flow." [Rodi-80]

Instantaneous variable ϕ ➡ $\phi = \overline{\phi} + \phi'$, $\overline{\phi} = \dfrac{1}{\Delta t}\displaystyle\int_{t-\frac{\Delta t}{2}}^{t+\frac{\Delta t}{2}} \phi(t)\, dt$

$\overline{\phi}$ ϕ'



t

$$\overline{\phi + \psi} = \overline{\phi} + \overline{\psi}$$

$$\overline{\partial_x \phi} = \partial_x \overline{\phi}$$

$$\overline{\partial_t \phi} = \partial_t \overline{\phi}$$

$$\overline{\overline{\phi}} = \overline{\phi}$$

$$\overline{\phi\, \overline{\psi}} = \overline{\phi}\, \overline{\psi}$$

$$\overline{\phi'} = 0$$

Turbulent Models

Medium values equations

Continuity equations:

$$\partial_i \overline{u}_i = 0$$

Navier - Stokes equations:

$$\rho \frac{\partial \overline{u}_i}{\partial t} + \rho \overline{u}_j \partial_j \overline{u}_i = -\partial_i \overline{p} + \partial_j \left[\mu \left(\partial_j \overline{u}_i + \partial_i \overline{u}_j \right) - \rho \overline{u'_i u'_j} \right] + \rho \overline{f}_i$$

$$\tau_{ij} = \overline{u'_i \, u'_j}$$

Reynolds stress tensor

Boussinesq

$$\tau_{ij} = \frac{2}{3} k \, \delta_{ij} - \frac{\mu^t}{\rho} \left(\partial_i \overline{u}_j + \partial_j \overline{u}_i \right)$$

$$k = \frac{1}{2} \overline{u'_i u'_i}$$

Turbulent kinetic energy

Turbulent Models

Models "ad-hoc" or zero-equation model or algebraic models: models for simple flows, which relate the turbulent viscosity with the velocity gradient through an empirical algebraic equation. The most commonly used in these models is the model of the Prandtl mixing length. These models are developed to solve certain flows and can not be extrapolated to the resolution of different flows.

Model of one equation: the eddy viscosity is related to a physical property representing the phenomenon of turbulence and use a partial differential equation of the type of transport to model this property. El más utilizado en este tipo de modelos es el modelo de la energía cinética turbulenta (k).

Turbulent Models

Model of two equations: the turbulent viscosity is calculated from the turbulent kinetic energy (k) and any other proxy for the phenomenon of turbulence. The behavior of k and the second variable is modeled by two partial differential equations of the type of transport. Some of these models are:

k-f f is the frequency energy of turbulent eddies

k-ω ω is the fluctuation in the time-averaged vorticity

k-ε ε is the dissipation rate of the turbulent kinetic energy

Turbulent Models

Reynolds stress model: do not use the definition of turbulent viscosity and require equations to model the terms $\overline{\underline{v}'\underline{v}'}$

Two-fluid models: are based on the concept of averaging the flow variables and turbulent flow considering non-turbulent.

Algebraic model – L model

The Reynolds stress tensor is related only to the mean flow distribution to each point. That is, implicitly assume that turbulence is generated and dissipated where there is no transport of turbulence.

Of the various algebraic models in the literature will be developed as the oldest example of them: the model of the Prandtl mixing length (1925).

Considering a shear flow in Cartesian coordinates with a single turbulence tension and a single component of the velocity gradient $\frac{\partial \bar{v}_x}{\partial y}$ significantly

Prandtl postulated that fluctuating movement characteristic velocity is equal to the average velocity gradient for the mixing length of movement (lm), established an analogy between molecular movement and the small eddies movement, postulating the existence of a length similar to the mean free path given by the kinetic theory, called the mixing length.

$$\hat{V} = l_m \left| \frac{\partial \bar{v}_x}{\partial y} \right|$$

$$\mu^t = \rho \, l_m^2 \left| \frac{\partial \bar{v}_x}{\partial y} \right|$$

Algebraic model – L model

Inconvenients $\mu^t = 0$; Channel center

However, there is turbulence in the area and μ^t must have a finite value.

The value taken for the *lm* length should be adjusted to each problem to obtain agreement with experiment. For example, in the vicinity of solid walls, the non-slip boundary conditions imply that mediaum velocity and the fluctuant velocity must be zero.

Reynolds stresses (as opposed to laminar tension) should be zero on the walls.

$$l_m = \kappa \; y$$

$$\tau = \tau_w = \mu^t \frac{d\overline{v}_x}{dy} = \rho \; \kappa^2 \; y^2 \left(\frac{d\overline{v}_x}{dy} \right)^2$$

Algebraic model – L model

Tube, R= radius, r = axis distance

$$l_m = 0.14 - 0.08\ (1 - r/R)^2 - 0.06\ (1 - r/R)^4$$

3D algebraic model

$$\mu^t = \rho\ l_m^2\ \left(2\ \overline{\underline{\underline{S}}} \cdot \overline{\underline{\underline{S}}}\right)^{\frac{1}{2}}$$

(k-L) model

The k-L model were developed to overcome the shortcomings of the hypothesis of mixing length

Abandon the direct relationship between the scale fluctuating motion velocity and mean velocity gradients and instead determine this scale from a transport equation.

The turbulent kinetic energy or kinetic energy due to fluctuations in the velocity (k) is a measure of the intensity of turbulence and √ k is a quantity that can characterize the scale of turbulent flow velocities.

Kolmogorov, Prandtl and Emmons independently proposed the following relation for the eddy viscosity

$$\mu^t = \rho C_\mu \sqrt{k}\, L$$

(k-L) model

$$\rho \frac{\partial k}{\partial t} + \rho \overline{u}_j \partial_j k = \partial_i \left[\left(\mu + \frac{\mu^t}{\sigma^k} \right) \frac{\partial k}{\partial x_i} \right] + P - \rho \frac{k^{3/2}}{L}$$

$$P = \mu^t \left(\partial_i \overline{u}_j + \partial_j \overline{u}_i \right) \partial_i \overline{u}_j = \rho \frac{k^{3/2}}{L}$$

$$\varepsilon = \frac{k^{3/2}}{L}$$

(k-ε) model

The trend in the development of turbulent flow models was considered models in which besides the scale model of fluctuating movement velocities also model the length scale from a transport equation, always within the Prandtl hypothesis.

Any relationship of the form $z = k^m L^n$ could be adopted as a variable independent, since *k* is known to solve their own transport equation.

The variable z is normally preferred dissipation rate of turbulent kinetic energy (ε), to be interpreted as the rate of turbulent energy transfer from large eddies to small eddies,

(k-ε) model

$$\rho \frac{\partial \varepsilon}{\partial t} + \rho \overline{u}_j \partial_j \varepsilon = \partial_i \left[\left(\mu + \frac{\mu^t}{\sigma^k} \right) \partial_i \varepsilon \right] + C_1 \frac{\varepsilon}{k} P - C_2 \rho \frac{\varepsilon^2}{k}$$

$C_1 = 1.44$; $C_2 = 1.92$; $C_\mu = 0.09$; $\sigma_k = 1.0$; $\sigma_\varepsilon = 1.3$

(k-ε) model

$$\nabla \cdot \mathbf{v} = 0$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} . \nabla \mathbf{v} - \nabla \cdot \left[\left(\mu + \mu^{t} \right) \left(\nabla \mathbf{v} + \nabla \mathbf{v}^{T} \right) \right] + \nabla P - \rho \mathbf{g} - \mathbf{F}_{b} - \mathbf{F}_{e} = \mathbf{0}$$

$$\rho \frac{\partial k}{\partial t} + \rho \mathbf{v} \cdot \nabla k - \nabla \cdot \left[\left(\mu + \frac{\mu^{t}}{\sigma_{k}} \right) \nabla k \right] - \mu^{t} \left(\nabla \mathbf{v} + \nabla \mathbf{v}^{T} \right) \mathbf{:} \nabla \mathbf{v} + \rho \frac{C_{\mu} k^{2}}{\mu^{t} / \rho} - \mathbf{F}^{k}{}_{b} = 0$$

$$\mu^{t} = C_{\mu} \rho \sqrt{k}\, L$$

$$\rho \frac{\partial \varepsilon}{\partial t} + \rho \mathbf{v} \cdot \nabla \varepsilon - \nabla \cdot \left[\left(\mu + \frac{\mu^{t}}{\sigma_{\varepsilon}} \right) \nabla \varepsilon \right] - \rho C_{\mu} C_{1} k \left(\nabla \mathbf{v} + \nabla \mathbf{v}^{T} \right) \mathbf{:} \nabla \mathbf{v} + \rho \frac{C_{2} \varepsilon^{2}}{k} - \mathbf{F}^{\varepsilon}{}_{b} = 0$$

$$L = \frac{k^{3/2}}{\varepsilon}$$

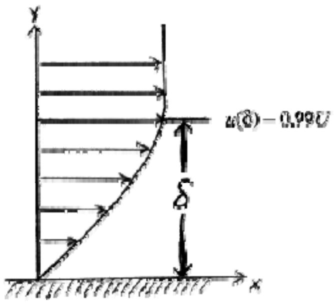
Walls

One of the most important problem of turbulence is when the local Reynolds turbulence number is low.

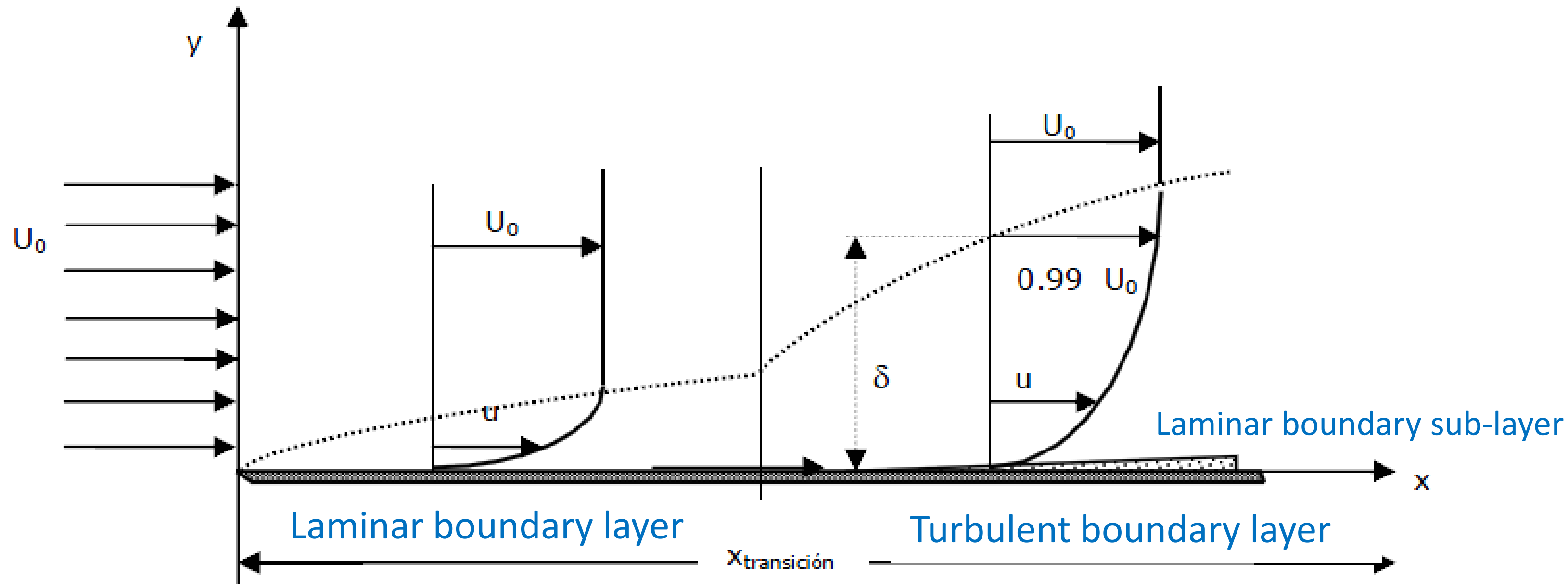
The presence of the wall ensures that over a finite region of the flow, however thin, the turbulence Reynolds number is low enough for molecular viscosity to influence directly the processes of production, destruction and transport of turbulence.

Two solutions for this problem:

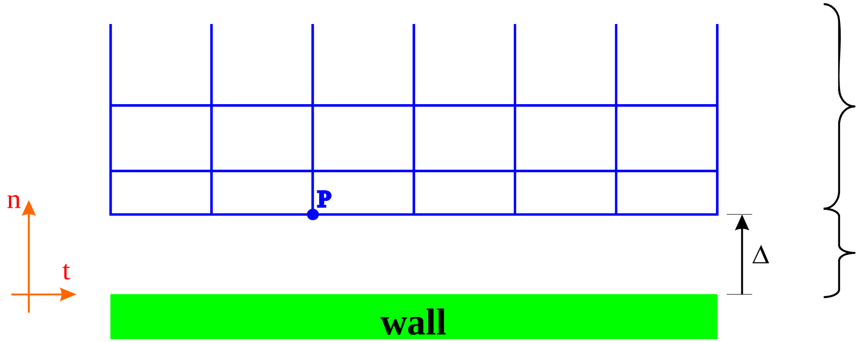
- Wall functions
- Low Reynolds number model



Walls


y
U₀
U₀
u
U₀
0.99 U₀
δ
u
Laminar boundary sub-layer
x
Laminar boundary layer
Turbulent boundary layer
xtransición

Walls



Area occupied by the finite element mesh

Area modeled by the wall functions

Wall function method

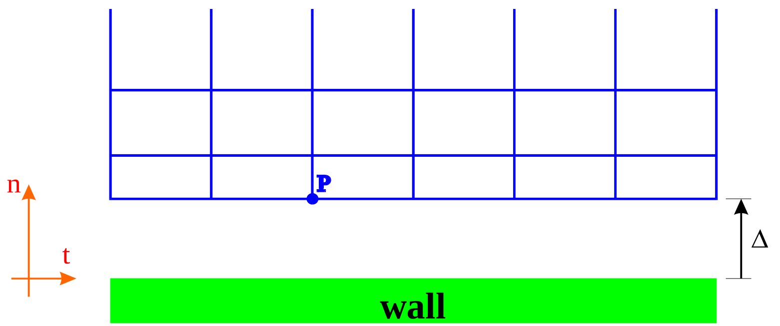
With the velocity (parallel to the wall) at the

$$u^+ = \begin{cases} y^+ & y^+ < 11.6 \\ \frac{1}{\kappa}\ln y^+ + 5.5 & y^+ > 11.6 \end{cases}$$

Calculate the friction velocity (u^*) in y = P

$$u^+ = \frac{u}{u_t} \qquad u_t = u^* = \sqrt{\frac{\tau_w}{\rho}} \qquad y^+ = \frac{\rho_w u_t y}{\mu_w} \qquad \kappa = 0.41$$

Walls



Area occupied by the finite element mesh

Area modeled by the wall functions

<u>Wall function method</u>

Apply in (i+1)-iteration the following boundary conditions at point P

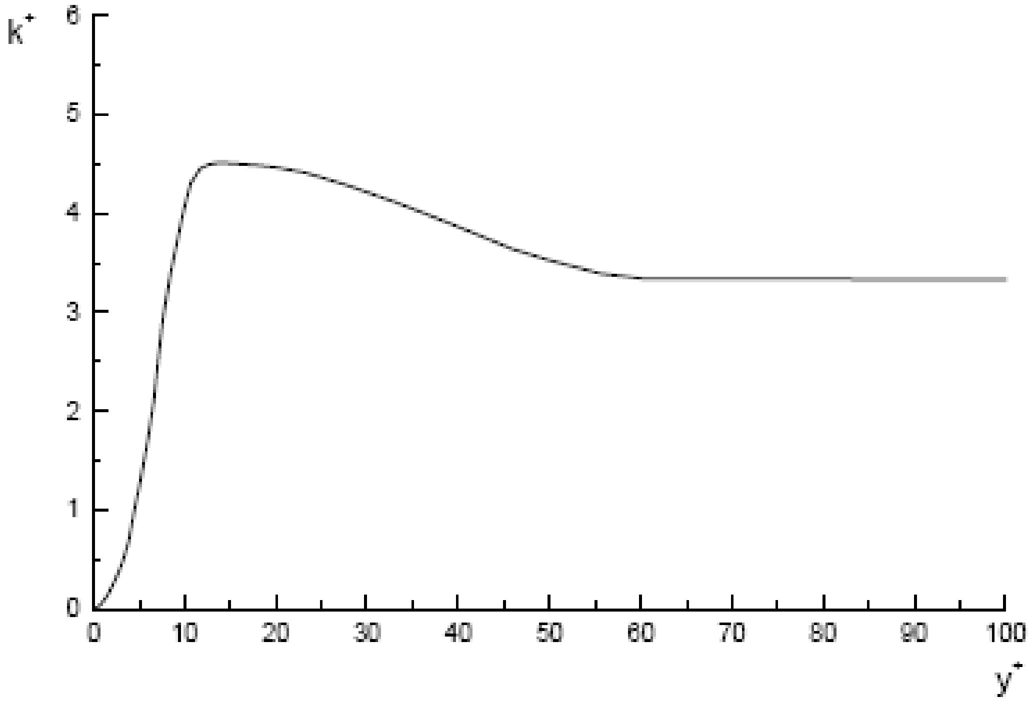
$$\tau = \rho u^{*2}$$

$$k = \frac{\tau}{\rho C_\mu^{1/2}} \quad \textbf{or} \quad \frac{dk}{dy} = 0$$

$$\varepsilon = \frac{C_\mu^{3/4} k^{3/2}}{\kappa y}$$

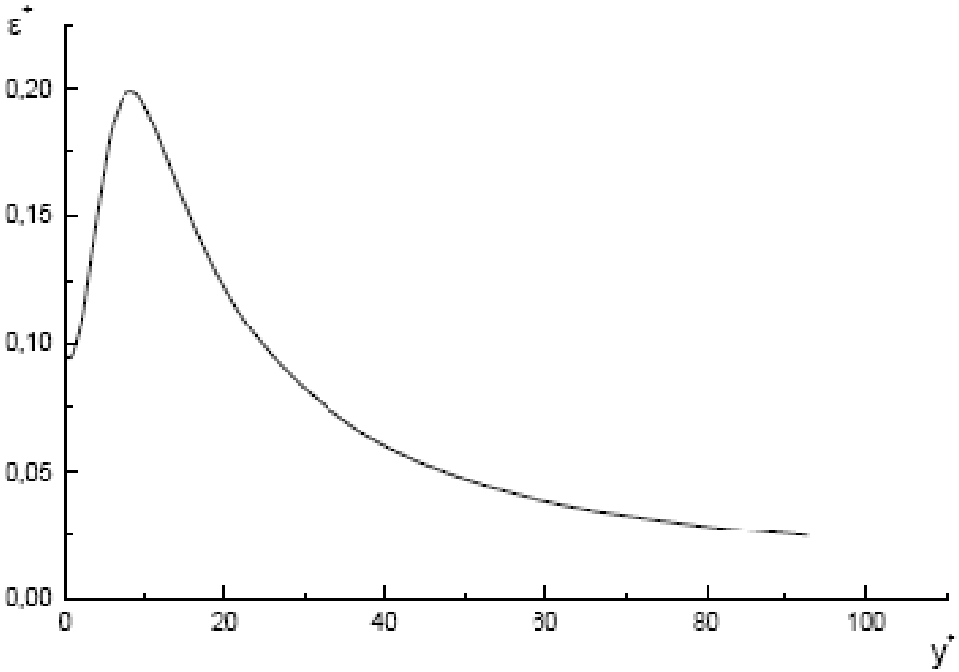
http://geolab.larc.nasa.gov/APPS/YPlus/

Walls



Turbulent kinetic energy

Walls



Dissipation rate of turbulent kinetic energy

Low Reynolds Number

1. Viscous diffusion of k and E must be included

2. Further terms must be added to account for the fact that the dissipation processes are not isotropic

$$\frac{Dk}{Dt} = \frac{\partial}{\partial y}\left(\mu + \frac{\mu_t}{\sigma_k}\frac{\partial k}{\partial y}\right) + \mu_t\left(\frac{\partial u}{\partial y}\right)^2 - \rho\,\varepsilon - 2\,\mu\cdot\left(\frac{\partial k^{\frac{1}{2}}}{\partial y}\right)^2$$

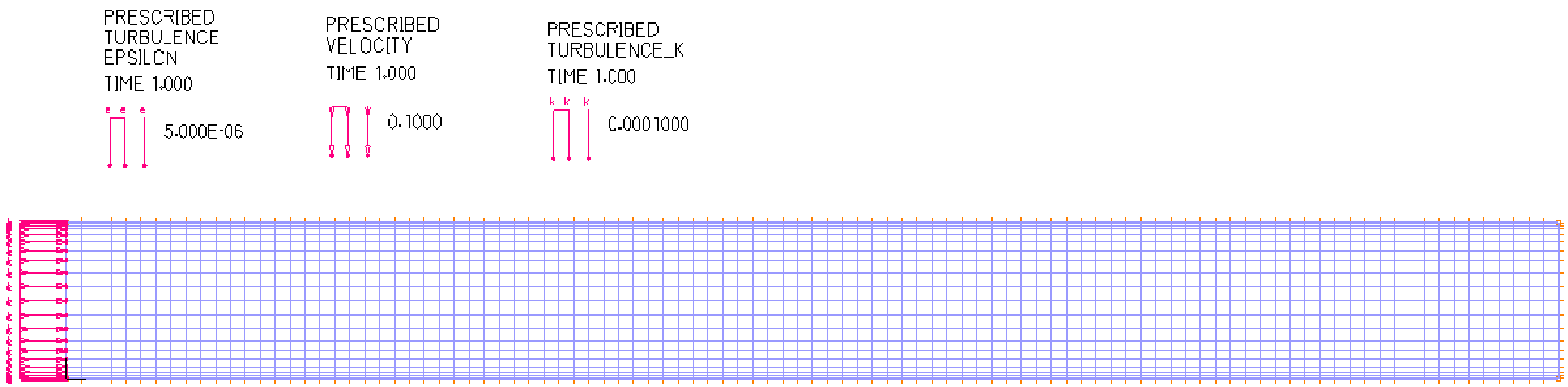
$$\frac{D\varepsilon}{Dt} = \frac{\partial}{\partial y}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y}\right) + c_1\,f_1\,\frac{\varepsilon}{k}\,\mu_t\left(\frac{\partial u}{\partial y}\right)^2 - c_2\,f_2\,\frac{\rho\,\varepsilon^2}{k} + 2.0\,\mu\,\mu_t\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$\mu_t = {c_\mu \cdot f_\mu \cdot \rho \cdot k^2}/{\varepsilon}$$

Channel

Boundary conditions at the entrance

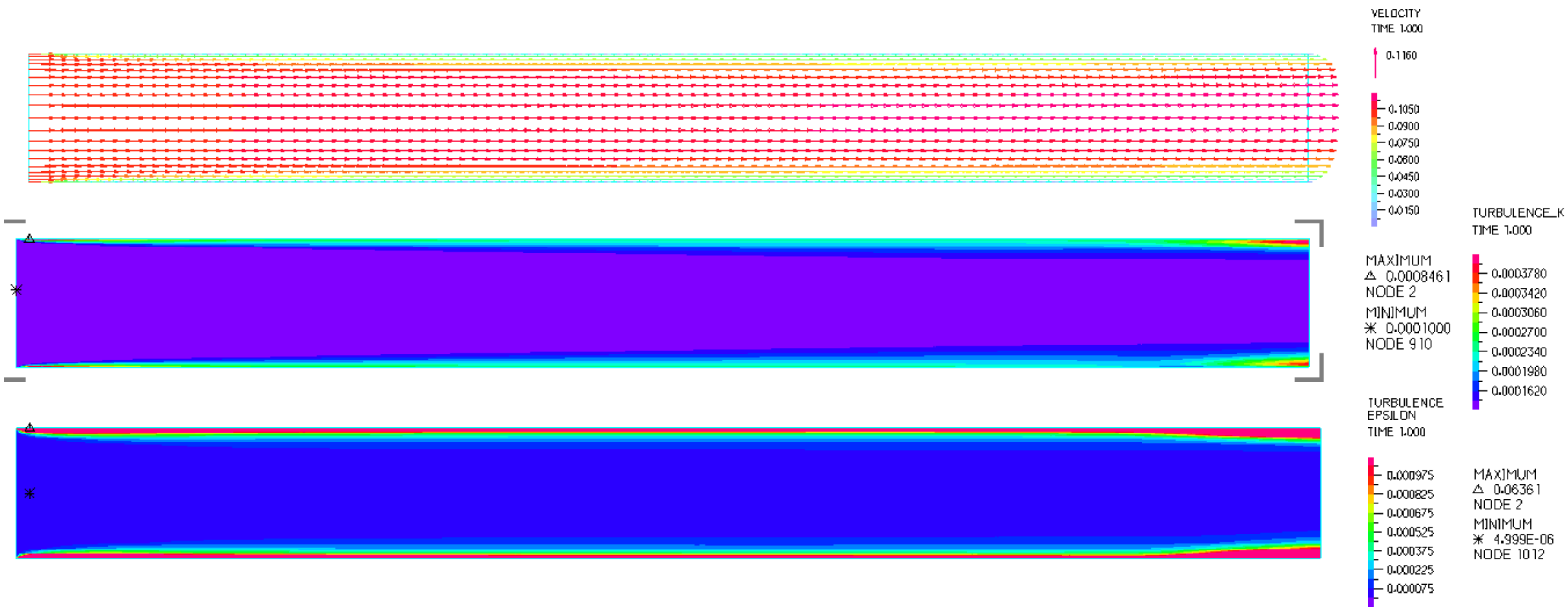
- Velocities
- Turbulent kinetic energy (k)
- Dissipation rate of turbulent kinetic energy (e)



Boundaty conditions at the walls:

- No slip

Channel


VELOCITY
TIME 1.000
0.1160
0.1050
0.0900
0.0750
0.0600
0.0450
0.0300
0.0150
TURBULENCE_K
TIME 1.000
MAXIMUM
△ 0.0008461
NODE 2
MINIMUM
✳ 0.0001000
NODE 910
0.0003780
0.0003420
0.0003060
0.0002700
0.0002340
0.0001980
0.0001620
TURBULENCE
EPSILON
TIME 1.000
0.000975
0.000825
0.000675
0.000525
0.000375
0.000225
0.000075
MAXIMUM
△ 0.06361
NODE 2
MINIMUM
✳ 4.999E-06
NODE 1012

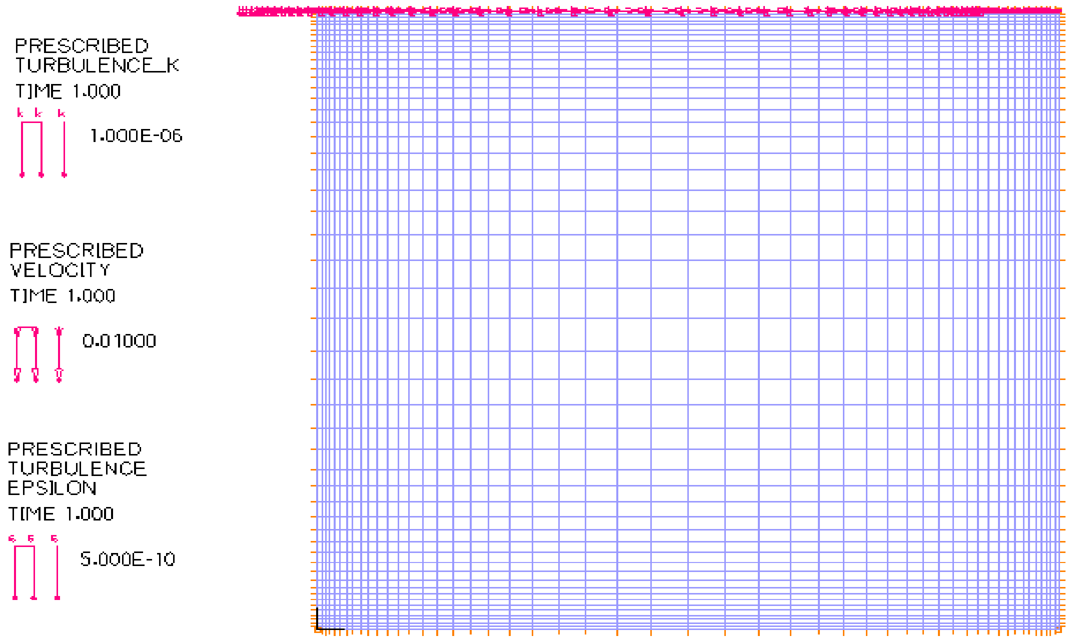
Cavity with sliding wall

Boundary conditions at the upper wall

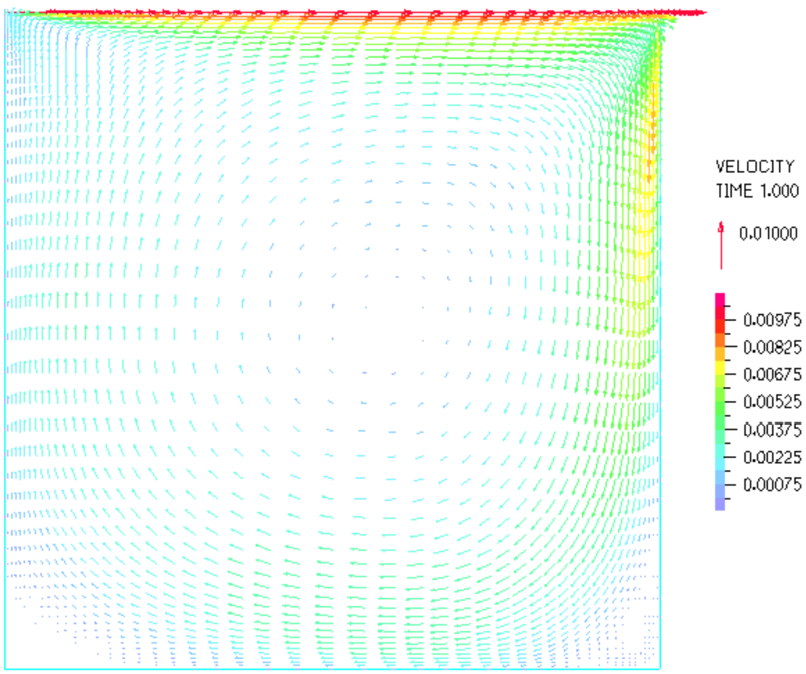
- Velocities
- Turbulent kinetic energy (k)
- Dissipation rate of turbulent kinetic energy (ε)

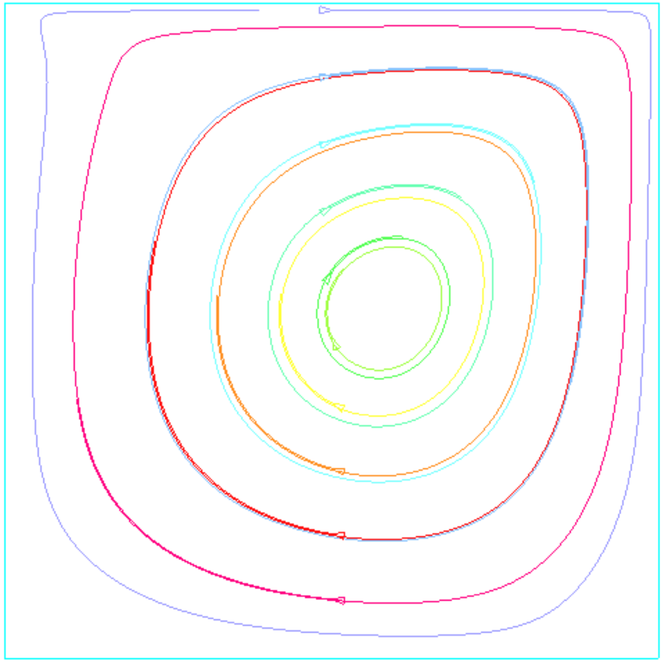
Boundary conditions at other walls

- No slip

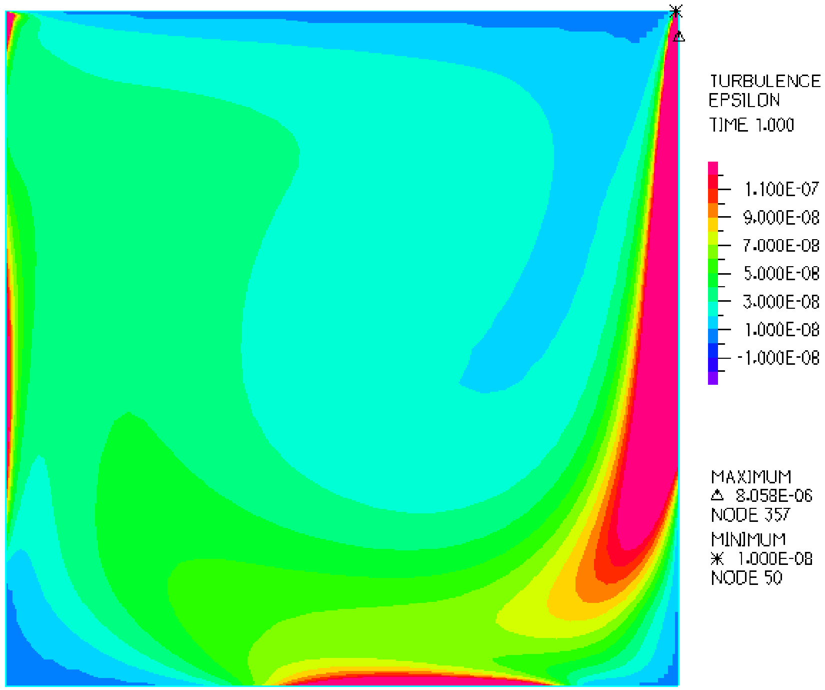


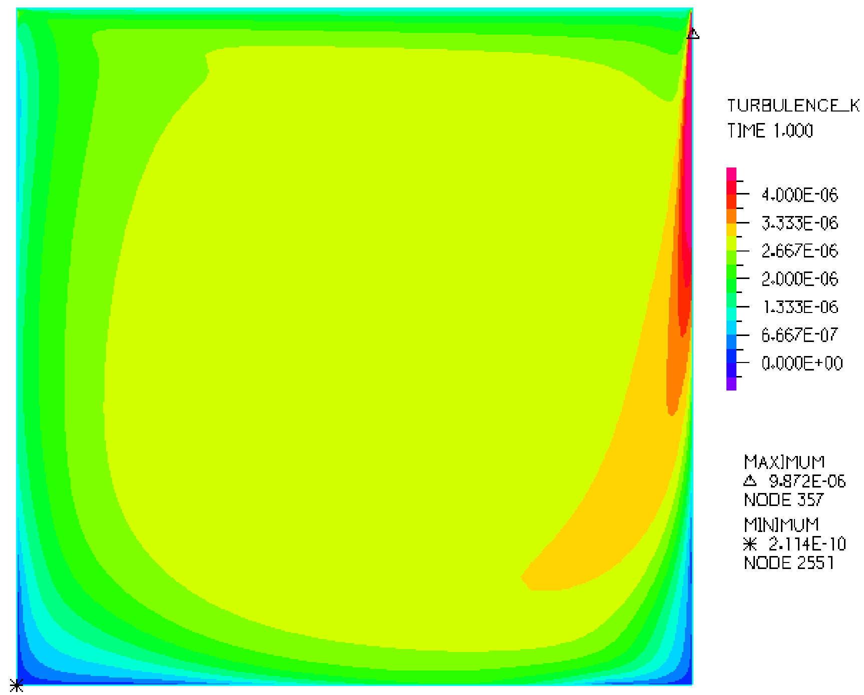
Cavity with sliding wall



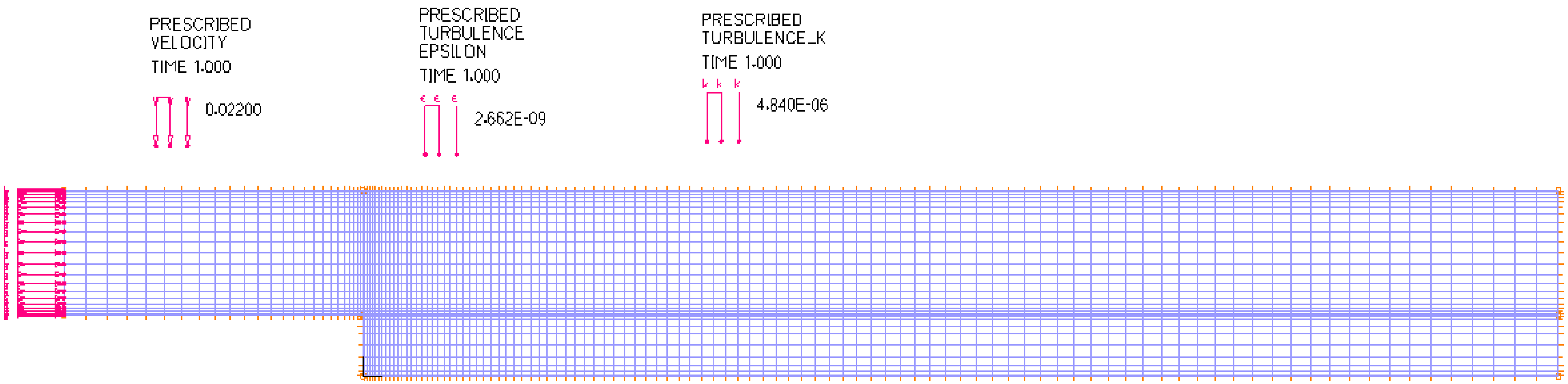


Cavity with sliding wall



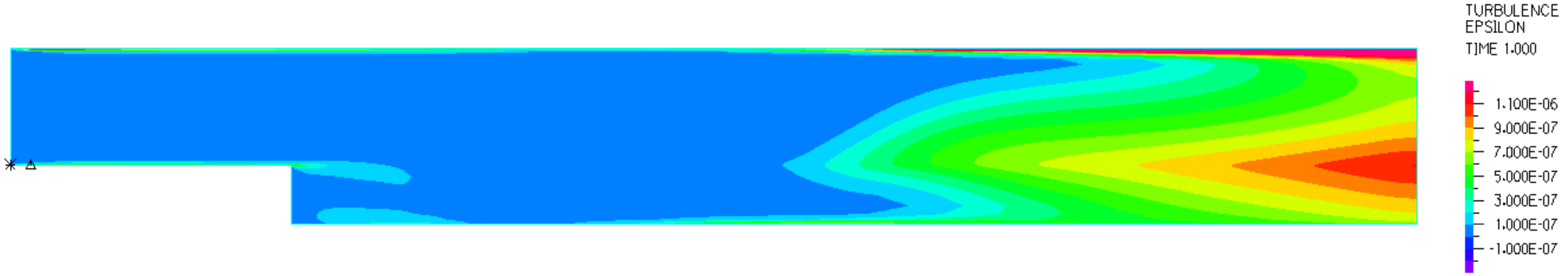


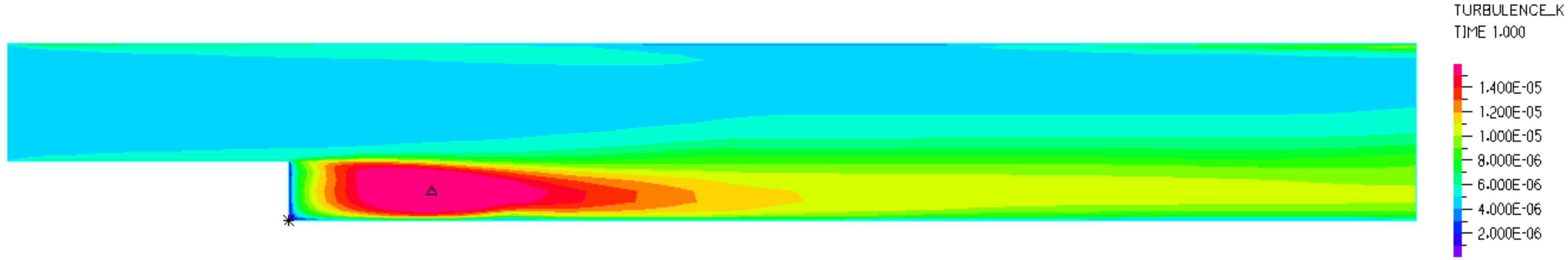
Backward facing step



Velocities = 0.022 m/s
Reynolds = 44000

Backward facing step





Backward facing step